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Eigenvalues and colorings of digraphs

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ABSTRACT

Wilf's eigenvalue upper bound on the chromatic number is extended to the setting of digraphs. The proof uses a generalization of Brooks' Theorem to digraph colorings.

1. Introduction

The purpose of this note is twofold. Firstly, a classical theorem of Wilf [11] providing an upper bound on the chromatic number of a graph in terms of its largest eigenvalue is extended to digraphs. The first impression might be that such an extension is impossible since every *transitive tournament* (this

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is an acyclic orientation of a complete graph) has all eigenvalues equal to zero. However, with the right definition of digraph colorings, as introduced by Neumann-Lara [10] and by Bokal et al. [2], this becomes reality. And this is also our second motivation – showing that the chromatic number of digraphs introduced in [2,10] is a natural concept with close ties to other branches of graph theory and combinatorics.

In this note we treat directed graphs (shortly *digraphs*). They are always assumed to be finite and *simple*, i.e. no loops and no multiple edges in the same direction are allowed. However, it is allowed to have two oppositely oriented edges joining the same pair of vertices. We use standard terminology and notation and refer to [1] for an extensive treatment of digraphs.

We use xy to denote the edge joining vertices x and y, where x is the *initial vertex* and y is the *terminal vertex* of the edge xy. For a digraph D and $v \in V(D)$, $e \in E(D)$, we denote by $D - v$ and $D - e$ the subdigraph of D obtained by deleting v and the subdigraph obtained by removing e, respectively. If $e = uv$ is not an edge of D, then we denote by $D + e$ the digraph obtained from D by adding the edge e. For a vertex $v \in V(D)$ we let $\deg_D^+(v)$ and $\deg_D^-(v)$ denote the *out-degree* (the number of edges whose initial vertex is v) and the *in-degree* (the number of edges whose terminal vertex is v) of v in D, respectively. We denote by $\Delta^+(D)$ and $\delta^+(D)$ the maximum and the minimum out-degree of D, respectively. Similarly, $\Delta^-(D)$ and $\delta^-(D)$ denote the maximum and the minimum in-degree of D. If D' is a subdigraph of D, we write $D' \subseteq D$. Every undirected graph G determines a *bidirected graph* $D(G)$ that is obtained from G by replacing each edge with two oppositely directed edges joining the same pair of vertices.

For a digraph D, we let $A(D)$ denote the *adjacency matrix* of D, and let $\rho(G)$ denote its *spectral radius*, the largest modulus of an eigenvalue of $A(D)$. It follows from the Perron–Frobenius theorem (see, e.g. [7]) that $\rho(D)$ is an eigenvalue of D and that there is a corresponding eigenvector whose coordinates are all non-negative. The spectra of undirected graphs are well treated in the literature, see [5,8], but there is not much known about digraphs. Recently, Brualdi wrote a stimulating survey on this topic [3], and we refer the reader to that article for additional information.

2. Dichromatic number and critical digraphs

Let D be a digraph. A vertex set $A \subseteq V(D)$ is *acyclic* if the induced subdigraph $D[A]$ is acyclic. A partition of $V(D)$ into k acyclic sets is called a *k-coloring* of D. The minimum integer k for which there exists a k-coloring of D is the *chromatic number* $\chi(D)$ of the digraph D.

The above definition of the chromatic number of a digraph was first introduced by Neumann-Lara [10]. The same notion was independently introduced much later by the author when considering the circular chromatic number of weighted (directed or undirected) graphs [9]. The chromatic number of digraphs was further investigated in [2].

Clearly, if G is an undirected graph, and D is the digraph obtained from G by replacing each edge with the pair of oppositely directed edges joining the same pair of vertices, then $\chi(D)$ is the same as the usual chromatic number of the undirected graph G since any two adjacent vertices in D induce a directed cycle of length two.

A digraph D is *strongly connected* if for every $x, y \in V(D)$ there exists a directed path from x to y and a directed path from y to x. Vertices of D can be partitioned into *strongly connected components* (the maximal strongly connected subdigraphs). Colorings of different strongly connected components cannot produce a monochromatic cycle. This implies that the study of colorings of digraphs can be restricted to strongly connected digraphs.

Lemma 2.1. *Let $D_1, \ldots, D_k$ be the strongly connected components of a digraph D. Then $\chi(D) = \max\{\chi(D_i) \mid 1 \leqslant i \leqslant k\}$.*

Suppose that $v \in V(D)$ is a vertex such that $\chi(D - v) < \chi(D)$. Then we say that v is a *critical vertex*. If every vertex of D is critical and $\chi(D) = k$, then we say that D is a *k-critical digraph*. Note that every digraph whose chromatic number is $\geqslant k$ contains an induced subgraph that is k-critical.

Lemma 2.2. *If v is a critical vertex in a digraph D whose chromatic number is k, then $\deg_D^+(v) \geqslant k - 1$ and $\deg_D^-(v) \geqslant k - 1$.*

Proof. If $\deg_D^+(v) \leqslant k-2$, then a $(k-1)$-coloring of $D-v$ uses at most $k-2$ colors on the out-neighbors of v. By setting the color of v to be one of the colors that are not used on the out-neighbors of v, we obtain a $(k-1)$-coloring of D, a contradiction. The proof for the in-degree bound is the same. $\square$

The following result is a generalization of the Brooks Theorem characterizing undirected graphs whose chromatic number is larger than the maximum degree.

Theorem 2.3. *Suppose that D is a k-critical digraph in which each vertex v satisfies* $\deg_D^+(v) = \deg_D^-(v) = k-1$. *Then one of the following cases occurs*:

(a) *$k = 2$ and D is a directed cycle of length $n \geqslant 2$.*
(b) *$k = 3$ and D is a bidirected cycle of odd length $n \geqslant 3$.*
(c) *D is a bidirected complete graph of order k.*

Proof. The proof is trivial for $k \leqslant 2$, so we will assume that $k \geqslant 3$. By Lemma 2.1, D is strongly connected. If D is a bidirected graph, then we have (b) or (c) by the Brooks Theorem. So we may assume that v_1v_2 is an edge of D such that $v_2v_1 \notin E(D)$. Let U be the set of all out-neighbors of v_1. Suppose that U induces the bidirected complete graph on $k-1$ vertices. Then we consider the set W of out-neighbors of v_2. Note that $W = U \cup \{z\} \setminus \{v_2\}$, where $z \neq v_1$. Since $k \geqslant 3$, $U \cap W$ is non-empty. The in-neighbors of every vertex in $U \cap W$ are v_1 and $k-2$ vertices in U, so z is not among them. This shows that the out-neighbors of v_2 do not induce a bidirected complete graph. This proves that there is a vertex u whose out-neighborhood contains two vertices u_1, u_2 such that $u_1u_2 \notin E(D)$. Let $n = |D|$ and set $u_n := u$.

Let us now consider the digraph $D' = D - u_n$ and observe that u_1, u_2 are both out-neighbors of u_n. Another observation is that u_n has an in-neighbor that is different from u_1, u_2. This implies that D' has a vertex $u_{n-1} \neq u_1, u_2$ whose outdegree is less than $k-1$ (if $n \geqslant 4$). We now repeat the process with the digraph obtained after removing the vertex u_{n-1}, etc. In the ith step ($i = n-1, n-2, \ldots, 3$), we find a vertex $u_i \neq u_1, u_2$ whose outdegree or indegree is less than $k-1$ in the current digraph $D_i = D - \{u_{i+1}, u_{i+2}, \ldots, u_n\}$. We end up with the vertex list $u_1, \ldots, u_n$ and now we color the vertices starting with u_1 and u_2, giving them both color 1. When we are about to color u_i, we already have a coloring of $u_1, \ldots, u_{i-1}$ and since u_i has in- or outdegree less than $k-1$, there is an available color that we can use on u_i thus obtaining a $(k-1)$-coloring of D_i. (If color t does not appear among the in-neighbors of u_i or among the out-neighbors of u_i, then giving that color to u_i cannot create a directed cycle.) This argument works all the way to u_n, but u_n has in- and outdegree equal to $k-1$. However, it has two out-neighbors, namely u_1 and u_2, of the same color. Therefore there is a color that does not occur among the out-neighbors of u_n, and we can use that color to get a $(k-1)$-coloring of D. This contradiction to k-criticality of D completes the proof. $\square$

Let us observe that graphs in (a)–(c) in Theorem 2.3 are k-critical and that cases (b) and (c) are the same as the cases from Brooks Theorem concerning colorings of undirected graphs.

3. The spectral radius of digraphs

In this section we review basic properties of the spectral radius. These results are well known (cf., e.g., [7]), but we include them for completeness. The basic property of the spectral radius is its monotonicity [7, Theorem 1.8.18].

Lemma 3.1. *If A is a non-negative matrix and $A \leqslant B$ (element-wise), then $\rho(A) \leqslant \rho(B)$.*

Corollary 3.2. *If $D' \subseteq D$, then $\rho(D') \leqslant \rho(D)$.*

If A is a symmetric matrix, then its spectral radius is equal to the numerical radius of A. However, this property does not hold for non-symmetric matrices. For non-negative matrices one can use the following property for a similar effect: If A is a non-negative matrix and $x \geqslant 0$ is a non-negative vector such that $Ax \geqslant \alpha x$ for some $\alpha \in \mathbb{R}$, then $\rho(A) \geqslant \alpha$. An easy consequence of this fact is the following.

Lemma 3.3. *For every digraph D, we have $\rho(D) \geqslant \min\{\deg_D^+(v) \mid v \in V(D)\}$ and $\rho(D) \geqslant \min\{\deg_D^-(v) \mid v \in V(D)\}$.*

A matrix A of order n is *irreducible* if $(A+I)^{n-1}$ has all components positive. For the adjacency matrix of a digraph, this is easily seen to be equivalent to the condition that the digraph is strongly connected. This is the reason that the spectral radius of digraphs can be studied on its strongly connected components.

Lemma 3.4. *Let $D_1, \ldots, D_k$ be the strongly connected components of a digraph D. Then $\rho(D) = \max\{\rho(D_i) \mid 1 \leqslant i \leqslant k\}$.*

If $A(D)$ is irreducible, then $\rho = \rho(D)$ is an eigenvalue of $A(D)$ whose algebraic and geometric multiplicity is one, and its right eigenvector x and its left eigenvector y are both positive, cf. [7, Theorem 8.4.4]. This fact is used in the Perron–Frobenius theory to derive the following stronger monotonicity property of the spectral radius.

Lemma 3.5. *Let D be a strongly connected digraph and let D' be a proper subdigraph of D. Then $\rho(D') < \rho(D)$.*

4. Chromatic number and eigenvalues

The following theorem is an analogue of the Wilf bound [11] for the usual chromatic number of the graph. Wilf [11] also characterized when equality occurs. For undirected graphs, one can use Brooks theorem, but in the case of digraphs we shall make use of the extension provided by Theorem 2.3.

Theorem 4.1. *Let D be a loopless digraph. Then*

$$\chi(D) \leqslant \rho(A(D)) + 1. \tag{1}$$

If D is strongly connected, then the equality holds in (1) if and only if D is one of the digraphs listed in cases (a)–(c) in Theorem 2.3 for $k = \chi(D)$.

Proof. By the monotonicity of the spectral radius, we may assume that D is simple. Let $k = \chi(D)$ and let us consider a k-critical subdigraph D' of D. Since the spectral radius is monotone, $\rho(D') \leqslant \rho(D)$, and it suffices to prove that $\rho(D') \geqslant k-1$. However, this is an easy consequence of Lemma 2.2 and Lemma 3.3.

To prove the second part of the theorem, let us observe that the above proof combined with Lemma 3.3 and Lemma 3.5 shows that equality occurs if and only if D is k-critical and the indegree and the outdegree of every vertex is equal to $k-1$. Now, Theorem 2.3 applies. □

Hoffman [6] and Cvetković [4] found lower bounds on the chromatic number of a graph expressed in terms of the eigenvalues of its adjacency matrix. It would be interesting to figure out if their bounds also apply to the setting of the chromatic number of digraphs.

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